SPRING 2025 MATH 590: QUIZ 1

Name:

1. Determine whether or not $\mathbb{R}^2$ with the usual addition, but with scalar multiplication given by: $\lambda \cdot (x, y) = (\lambda x, \frac{1}{\lambda} y)$, if $\lambda \neq 0$ and $(0, 0)$, if $\lambda = 0$, is a vector space. You must justify your answer. (4 points)

Solution. This does not give a vector space, as the distributive property for scalar multiplication fails. In other words, for $a, b \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$, it need not hold that $(a + b) \cdot (x, y) = a \cdot (x, y) + b \cdot (x, y)$. For example, take $a = 1, b = 2$. Then $(1+3) \cdot (x, y) = (3x, \frac{y}{3})$. But $1 \cdot (x, y) + 2 \cdot (x, y) = (x, y) + (2x, \frac{y}{2}) = (3x, \frac{3y}{2})$.

2. For the vector space V and subspaces $W_1, W_2 \subseteq V$, define what it means for V to be the *direct sum* of W_1 and W_2. Then show that if $V = \mathrm{M}_{2\times 2}(\mathbb{R})$, W_1 is the space of 2×2 symmetric matrices and W_2 is the space of 2×2 skew-symmetric matrices, then $V = W_1 \oplus W_2$. Note: A is symmetric if $A^t = A$ and A is skew-symmetric of $A^t = -A$. (6 points)

Solution. For the first statement, V is the direct sum of W_1 and W_2 if: (i) $V = W_1 + W_2$, i.e., any $v \in V$ can be written as $v = w_1 + w_2$, for some $w_1 \in W_1$ and $w_2 \in W_2$ and (ii) $W_1 \cap W_2 = \{\vec{0}\}$.

For the second statement, given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we can write $A = \begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{pmatrix} + \begin{pmatrix} 0 & \frac{b-c}{2} \\ \frac{c-b}{2} & 0 \end{pmatrix}$, where the first matrix in the sum belongs to W_1 and the second matrix in the sum belongs to W_2. Thus, $\mathrm{M}_{2\times 2}(\mathbb{R}) = W_1 + W_2$. Now suppose the symmetric matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is also skew-symmetric. Then, $A = A^t = -A$, i.e., $A = -A$. Thus, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, showing $W_1 \cap W_2$ is zero. Therefore, $\mathrm{M}_{2\times 2}(\mathbb{R}) = W_1 \bigoplus W_2$.